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The Universal Covers of a Family of Extended Generalized Quadrangles

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A sufficient condition for the simple connectedness is given for an infinite family of extended generalized quadrangles $\mathcal{Y}(\mathcal{S})$ of order $(q+1, q-1)$ constructed in [7] from a family $\mathcal{S}$ of planes in $PG(5, q)$ with some conditions. Applying this, $\mathcal{Y}(\mathcal{S})$ is shown to be simply connected when $\mathcal{S}$ is obtained from a $(q+1)$-arc in $PG(3, q)$ except for $q = 4$, and when $\mathcal{S}$ is constructed from the hyperovals for which the associated permutation polynomials are explicitly given in the list [6, p.299], except possibly for a class of Payne.

1. INTRODUCTION

In [7], an infinite family of extended generalized quadrangles $\mathcal{Y}(\mathcal{S})$ of order $(q+1, q-1)$ is constructed from a family of planes $\mathcal{S}$ in $PG(5, q)$ $(q = 2^e)$ with some conditions (see 2.1 for the precise statements). Though there is no classification of such families $\mathcal{S}$ at the present time, this paper provides a sufficient condition for $\mathcal{Y}(\mathcal{S})$ to be simply connected in terms of the additive subgroup of $\mathbf{F}_q$ generated by the values of some function (Theorem 1). That condition is naturally obtained from the fact that the fundamental group of $\mathcal{Y}(\mathcal{S})$ is generated by quadrangles with some specified antipodal point (Section 3, Lemma 7), and from the investigation of the connected components of the common neighbor of the base point and the antipodal point (Lemma 3).

There are two known constructions of a family $\mathcal{S}$ as above: one, due to Thas [5], from a $(q+1)$-arc K of $PG(3, q)$; the other one, due to the author [7] from an arbitrary hyperoval O of $PG(2, q)$. We denote those two families by $\mathcal{S}(K)$ and $\mathcal{S}(O)$, respectively. We will find a test for the simple connectedness of $\mathcal{Y}(\mathcal{S})$, relying on calculations of the values of a function β_{124}, to be defined in Theorem 1 (Section 3). Exploiting the classification of $(q+1)$-arcs [2], we will see that, when $\mathcal{S} = \mathcal{S}(K)$, $\beta_{124}(t)^{-1} = t^{2^m+1}$ for some m (Section 4). As for $\mathcal{S} = \mathcal{S}(O)$, once the explicit shape of the permutation polynomial f determining the hyperoval $O = O(f)$ is known, the function $\beta_{124}(t)$ is determined by the equality $\beta_{124}(t)^{-1} = tf(t)$ (Section 5). According to Theorem 1 of this paper, $\mathcal{Y}(\mathcal{S})$ is simply connected if $\mathbf{F}_q$ is generated by the values of β_{124}. The latter condition is easy to check when f is as in the list of [6, p.299], except when f belongs to the class of Payne (Lemmas 8 and 9).

Thus, for every $(q+1)$-arc K and every hyperoval $O = O(f)$ with f as in [6, p.299], the extended generalized quadrangle $\mathcal{Y}(\mathcal{S})$, with $\mathcal{S} = \mathcal{S}(K)$ or $\mathcal{S} = \mathcal{S}(O)$, is simply connected except when $q = 4$ and except possibly when O is a hyperoval of Payne (Theorems 2 and 3). The case $q = 4$ has been analyzed earlier by the author; in this case $\mathcal{Y}(\mathcal{S})$ admits a simply connected double cover ([8, 1.1]).

As for the basic terminology on geometry (especially that related with the fundamental group), we follow the book of Pasini ([4], in particular Chapter 12).

2. SOME INDUCED SUBGRAPHS

2.1. The EGQs $\mathcal{Y}(\mathcal{S})$ of order $(q+1, q-1)$. Throughout the paper, we consider a geometry $\mathcal{Y}(\mathcal{S})$ constructed as in [7, Section 2] from a family $\mathcal{S} = \{X_1, \ldots, X_{q+3}\}$ of $q+3$ projective planes of $PG(5, q)$ with the following properties:

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(1) $X_{ij} := X_i \cap X_j$ is a projective point for every $i \neq j \in I := \{1, \dots, q+3\}$.
(2) For each $i \in I$, the set $O_i := \{X_{ij} \mid j \in I - \{i\}\}$ of $q+2$ points forms a hyperoval in the plane X_i.
(3) $\mathcal{S}$ generates $PG(5, q)$.

As for the condition (3), note that Thas [5] constructed a geometry $\mathcal{T}(\mathcal{S}')$ from some family $\mathcal{S}'$ of planes satisfying the conditions (1),(2), but not (3). For that family $\mathcal{S}'$ in [5], he also showed that $\mathcal{T}(\mathcal{S}')$ is a quotient of a geometry $\mathcal{Y}(\mathcal{S})$ for some family $\mathcal{S}$ of planes with the conditions (1),(2) and (3).

Recall that, under the embedding of $PG(5, q)$ into $PG(6, q)$, the set of points, lines and planes of $\mathcal{Y}(\mathcal{S})$ are defined to be the set of subspaces of $PG(6, q)$ of the shapes $\langle f, X_i\rangle$, $\langle f, X_{ij}\rangle$ and $[f]$, respectively, for $X_i \in \mathcal{S}$ and every point $[f]$ of $PG(6, q)$ outside $PG(5, q)$. (In this paper $\langle X\rangle$ denotes the span of a set of vectors X. For a vector x, we use the symbol $[x] = \langle\{x\}\rangle$.) Incidence inherited from $PG(5, q)$. The plane X_i in a point $P = \langle f, X_i\rangle$ is referred to as the *base* of P. The resulting geometry $\mathcal{Y}(\mathcal{S})$ is an extended generalized quadrangle of order $(q+1, q-1)$, in which the residue at a point of the form $\langle f, X_i\rangle$ is the dual of the Tits generalized quadrangle $T^*(O_i)$ $(i \in I)$ [7, Lemma 2.3].

Two constructions are known for a family $\mathcal{S}$ of planes with the conditions (1),(2),(3) above. According to one of these constructions, $\mathcal{S}$ is a set of $q+2$ planes obtained by applying the Veronesean map to the lines on the dual O^* of an arbitrary hyperoval O in $PG(2, q)$, together with the nucleus plane N of the Veronesean variety [7]. This family will be denoted by $\mathcal{S}(O)$. In the extended generalized quadrangle $\mathcal{Y}_q(O) := \mathcal{Y}(\mathcal{S}(O))$, the residue at a point $\langle f, X\rangle$ for $X \neq N$ is the dual of $T^*(O')$ for a regular (classical) hyperoval O', but that for $X = N$ is the dual of $T^*(O)$ with O as above.

According to the other construction, $\mathcal{S}$ is a set of $q+1$ planes obtained by applying the Klein correspondence to the points on a $(q+1)$-arc K in $PG(3, q)$ together with the two systems of generators of K [5, Section 3]. Note that K is projectively equivalent to $\{[1, t, t^{2^m}, t^{2^m+1}] | t \in \mathbf{F}_q\} \cup \{[0, 0, 0, 1]\}$ for some $1 \leq m \leq e$ prime to e, where $q = 2^e$. This family will be denoted by $\mathcal{S}(K)$. In the extended generalized quadrangle $\mathcal{Y}_q(K) := \mathcal{Y}(\mathcal{S}(K))$, the residue at a point $\langle f, X\rangle$ for X distinct from the images of two systems of generators is isomorphic to the dual of $T^*(O_m)$, where $O_m := \{[1, t, t^{2^m}] \mid t \in GF(q)\} \cup \{[0, 0, 1], [0, 1, 0]\}$. The residues at $\langle f, X\rangle$ with X corresponding to the two systems of generators of K are isomorphic to the dual of $T^*(O')$ for a regular hyperoval O'.

These two constructions of $\mathcal{S}(O)$ and $\mathcal{S}(K)$ coincide exactly when O is regular and K is a twisted cubic (that is, $m = 1$ or $e-1$). In this case, $\mathcal{Y}_q(O) \cong \mathcal{Y}_q(K)$, and the residue at every point is isomorphic to the dual of $T_2^*(O)$.

2.2. The collinearity graph of $\mathcal{Y}(\mathcal{S})$. Let $\mathcal{Y}(\mathcal{S})$ be the extended generalized quadrangle constructed from a family $\mathcal{S}$ of planes satisfying the conditions (1)–(3) in Subsection 2.1. Choose an index set $I = \{1, \dots, q+3\}$ for the family $\mathcal{S}$ of planes, and fix $P_0 = \langle f, X_1\rangle$ as the base point of $\mathcal{G}$. We take a base x_{ij} for each projective point $X_{ij} = X_i \cap X_j$ $(i \neq j \in I)$.

Two points $\langle f, X_i\rangle$ and $\langle f', X_j\rangle$ are collinear if and only if $i \neq j$ and $f + f' \in \langle X_i, X_j\rangle$. In particular, as $X_i \cap X_j$ is a point, $\langle X_i, X_j\rangle$ is a hyperplane of $PG(5, q)$ for $i \neq j \in I$. Moreover, $X_i \cap \langle X_j, X_k\rangle = \langle X_{ij}, X_{ik}\rangle$ for mutually distinct indices $i, j, k \in I$, because the intersection of X_i with $\langle X_j, X_k\rangle$ should be a projective line, as $X_i \not\subseteq \langle X_j, X_k\rangle$ by the condition (3) in Subsection 2.1.

These simple remarks will be frequently used later without further references.

The collinearity graph Γ of $\mathcal{G}$ is examined in [7, Section 5]. (Note that the arguments there do not depend on the choice of $\mathcal{S}$.) We follow the usual convention for graphs. The diameter of Γ is three, and the set $\Gamma_1 := \Gamma(P_0)$ of points collinear with P_0 consists of the $(q+2)q^2$ points of the form $\langle f + x_1, X_i\rangle$ for $i \in I$ and $x_1 \in X_1 - X_{1i}$. The set $\Gamma_3 := \Gamma_3(P_0)$ at distance 3 from P_0 consists of $q(q-1)^2/2$ points of the form $\langle f + x, X_1\rangle$ for every $x \in PG(5, q)$ not contained in $\cup_{i \in I - \{1\}}\langle X_i, X_1\rangle$.

On the other hand, the set $\Gamma_2(P_0)$ of points at distance two from P_0 is divided into two subsets Γ_2^1 and Γ_2^2 [7, Lemma 5.1]:

$$\Gamma_2^1 := \{\langle f + x_i, X_1\rangle \mid x_i \in X_i - X_{1i}, 1 \neq i \in I\},$$
$$\Gamma_2^2 := \{\langle f + x_1 + x_i, X_j\rangle \mid x_1 \in X_1, x_i \in X_i, i \neq j \in I - \{1\}\}.$$

Notice that a point $R = \langle f + x_1 + x_i, X_j\rangle$ belongs to Γ_2^2 only if $x_i \notin \langle X_1, X_j\rangle$: for, because $\Gamma_1 = \{\langle f + x_1, X_i\rangle | x_1 \in X_1 - X_{1i}, 1 \neq i \in I\}$ does not contain R, there are no elements $y_1 \in X_1, y_j \in X_j$ such that $f + x_1 + x_i + y_j = f + y_1$, and hence $x_i \notin \langle X_1, X_j\rangle$.

By [7, Figure 1]), there is no edge in Γ between Γ_2^1 and Γ_3.

2.3. The common neighbor $\Gamma_1 \cap \Gamma(R)$ for $R \in \Gamma_2(P_0)$. Choose a point $R' = \langle f + x_a, X_1\rangle$ in Γ_2^1 ($x_a \in X_a - X_{1a}$, $1 \neq a \in I$). In the plane X_a, the line through two distinct points $[x_a]$ and X_{1a} intersects a hyperoval $O_a = \{X_{ac} | c \in I - \{a\}\}$ at exactly two points X_{1a} and X_{ab}, say. Thus, there is a unique index $b \in I - \{1, a\}$ such that $x_a \in \langle X_{1a}, X_{ab}\rangle$. With this notation, [7, Lemma 5.2(2)] says that the set $\Gamma(P_0) \cap \Gamma(R')$ consists of the following $2q^2$ points for $x_1 \in X_1$:

$$Q'_a(x_1) := \langle f + x_1, X_a\rangle, \quad Q'_b(x_1) := \langle f + x_a + x_1, X_b\rangle.$$

Note that $Q'_a(x_1) = Q'_a(y_1)$ (resp. $Q'_b(x_1) = Q'_b(y_1)$) holds if and only if $x_1 + y_1 \in X_{1a}$ (resp. $\in X_{1b}$). Furthermore, $\{Q'_a(x_1) | x_1 \in X_1\}$ and $\{Q'_b(x_1) | x_1 \in X_1\}$ are cocliques in the induced subgraph of Γ on $\Gamma(P_0) \cap \Gamma(R')$. The following is easily verified.

LEMMA 1. *Two points $Q'_a(x_1)$ and $Q'_b(y_1)$ are collinear if and only if $x_1 + y_1$ lies in the projective line $\langle X_{1a}, X_{1b}\rangle$ of X_1.*

Choose now a point $R = \langle f + x_1 + x_a, X_b\rangle$ ($a \neq b \in I - \{1\}$, $x_a \in X_a$) in Γ_2^2. For each index $i \in I - \{1, a, b\}$, the three distinct projective points X_{1a}, X_{ab}, X_{ai} lie on the hyperoval $O_a := \{X_{aj} | j \in I - \{a\}\}$ of X_a, and hence their bases x_{1a}, x_{ab}, x_{ai} are linearly independent. We write $x_a \in X_a$ as a linear combination of these vectors:

$$x_a = \alpha_{1a}(i)x_{1a} + \alpha_{ab}(i)x_{ab} + \alpha_{ai}x_{ai},$$

where $\alpha_{1a}(i), \alpha_{ab}(i), \alpha_{ai}$ are elements of $\mathbf{F}_q$ uniquely determined by i (under the choice of the indices $1, a, b$, the basis x_{ij} for $i, j \in \{1, a, b\}$ and x_a).

With the notation above, [7, Lemma 5.2(4)] says that the set $\Gamma(P_0) \cap \Gamma(R)$ consists of the following $q(q+1)$ points for $\alpha \in \mathbf{F}_q$ and $i \in I - \{1, a, b\}$:

$$Q_a(\alpha) := \langle f + \alpha x_{1b}, X_a\rangle, \quad Q_i(\alpha) := \langle f + \alpha_{1a}(i)x_{1a} + \alpha x_{1b}, X_i\rangle.$$

The following is easy to see.

LEMMA 2.

(1) *Two points $Q_a(\alpha)$ and $Q_i(\beta)$ ($i \in I - \{1, a, b\}$) are collinear if and only if $\alpha = \beta$.*
(2) *For $i \neq j \in I - \{1, a, b\}$, a point $Q_i(\alpha)$ is collinear with $Q_j(\beta)$ if and only if the projective point $[(\alpha_{1a}(i) + \alpha_{1a}(j))x_{1a} + (\alpha + \beta)x_{1b}]$ coincides with the point of intersection of two projective lines $\langle X_{1a}, X_{1b}\rangle$ and $\langle X_{1i}, X_{1j}\rangle$ of the projective plane X_1.*
In particular, given $Q_i(\alpha)$ and $j \in I - \{1, a, b, i\}$, there is a unique $\beta \in \mathbf{F}_q$ such that $Q_j(\beta)$ is collinear with $Q_i(\alpha)$.

2.4. Connected components of $\Gamma_1 \cap \Gamma(R)$ *for* $R \in \Gamma_2^2$. We keep the notation of Subsection 2.3 and Lemma 2.

We firstly observe that the function α_{1a} from $I - \{1, a, b\}$ to $\mathbf{F}_q$ sending each index i to $\alpha_{1a}(i)$ (see the expression for x_a in Subsection 2.3) is an injection. Suppose $\alpha_{1a}(i) = \alpha_{1a}(j)$ for some distinct i, j. Then we have $0 = x_a + x_a = (\alpha_{ab}(i) + \alpha_{ab}(j))x_{ab} + \alpha_{ai}x_{ai} + \alpha_{aj}x_{aj}$. As the three points X_{ab}, X_{ai} and X_{aj} lie on the hyperoval O_a in X_a, this implies that $\alpha_{ab}(i) + \alpha_{ab}(j) = \alpha_{ai} = \alpha_{aj} = 0$. Then $x_a \in \langle X_1, X_b \rangle$, which is against $R \in \Gamma_2^2$ (see the remark following the description of Γ_2^2 in Subsection 2.2). Thus the function α_{1a} is a bijection from $I - \{1, a, b\}$ onto $\mathbf{F}_q$.

In particular, there is a unique index, say c, of $I - \{1, a, b\}$ with $\alpha_{1a}(c) = 0$. Then $x_a = \alpha_{ab}x_{ab} + \alpha_{ac}x_{ac}$. It follows from Lemma 2(2) that for each $t \in I - \{1, a, b, c\}$ there is a unique value β of $\mathbf{F}_q$ such that $Q_c(0)$ is collinear with $Q_t(\beta)$. When we fix $1, a, b$ and hence c, this value β is uniquely determined by t as the element of $\mathbf{F}_q$ with

$$[\alpha_{1a}(t)x_{1a} + \beta x_{1b}] = \langle X_{1a}, X_{1b} \rangle \cap \langle X_{1c}, X_{1t} \rangle.$$

We denote by β_{1ab} the function sending each t to this $\beta \in \mathbf{F}_q$. It is immediate to see that β_{1ab} is a map from $I - \{1, a, b, c\}$ to $\mathbf{F}_q^{\times}$.

LEMMA 3. *With the notation above, if the additive group* $\mathbf{F}_q$ *is generated by* $\{\beta_{1ab}(t) | t \in I - \{1, a, b, c\}\}$, *then the induced subgraph* $\Gamma_1 \cap \Gamma(R)$ *is connected.*

PROOF. For each $\alpha \in \mathbf{F}_q$, let $C(\alpha)$ be the connected component of $\Gamma_1 \cap \Gamma(R)$ containing the point $Q_a(\alpha)$. By Lemma 2(1), $C(\alpha)$ contains the points $Q_i(\alpha)$ for all $i \in I - \{1, a, b\}$.

Note that $Q_i(\alpha)$ is collinear with $Q_j(\beta)$ iff $(f + \alpha_{1a}(i)x_{1a} + \alpha x_{1b}) + (f + \alpha_{1a}(j)x_{1a} + \beta x_{1b})$ lies in $\langle X_i, X_j \rangle$ iff $Q_i(\alpha + \gamma)$ is collinear with $Q_j(\beta + \gamma)$ for every $\gamma \in \mathbf{F}_q$ ($i \neq j \in I - \{1, a, b\}$). In particular, $Q_i(\beta)$ lies in the component $C(\alpha)$ iff $Q_i(\beta + \alpha)$ lies in the component $C(0)$. Thus, once we have determined the connected component $C(0)$, then the other components are obtained by 'shifting' $C(0)$.

Now the component $C(0)$ contains $Q_a(0)$ and $Q_c(0)$, and hence $Q_t(\beta_{1ab}(t))$ for all $t \in I - \{1, a, b, c\}$ by the definition of the function β_{1ab}. Thus $C(0) = C(\beta_{1ab}(t))$ for all $t \in I - \{1, a, b, c\}$. By Lemma 2(1), the $Q_a(\beta_{1ab}(t))$'s lie in $C(0)$, and therefore $Q_t(\beta_{1ab}(t) + \beta_{1ab}(s))$ lies in $C(0)$ for all $t, s \in I - \{1, a, b, c\}$ by the remark in the above paragraph. Hence, $C(0) = C(\beta_{1ab}(t) + \beta_{1ab}(s))$. Continuing this, we conclude that $C(0) = C(\alpha)$ for every $\alpha \in A$, where A is the set of the sums of elements of $\{\beta_{1ab}(t) | t \in I - \{1, a, b, c\}\}$. Thus, if A coincides with $\mathbf{F}_q$, the subgraph $\Gamma_1 \cap \Gamma(R)$ is connected. □

3. CLOSED PATHS IN THE COLLINEARITY GRAPH

We use the conventions stated in the previous section. Remark that the points and edges of each triangle in Γ are incident to a plane in common by [7, Lemma 2.5]. Thus:

LEMMA 4. *Each triangle in* Γ *is null-homotopic.*

As Γ is of diameter 3, each closed path in Γ at P_0 can be decomposed as a sum of triangles, quadrangles, pentagons, hexagons or heptagons. We may assume that each quadrangle or pentagon (resp. hexagon or heptagon) contains a point at distance 2 (resp. 3) from P_0, for otherwise it can be decomposed as the sum of polygons with shorter girth. Then no heptagon appears, because every point at distance 3 from P_0 has the same base X_1 and hence no two of them are collinear.

By similar observations for bases, a hexagon at P_0 is a sequence $(P_0, Q_1, R_2, S_3, R_4, Q_5, P_0)$ of six subsequently collinear points with $Q_i \in \Gamma_1$ $(i = 1, 5)$, $R_j \in \Gamma_2^2$ $(j = 2, 4)$, and $S_3 \in \Gamma_3$. Similarly, a pentagon at P_0 is a sequence of five subsequently collinear points $(P_0, Q_1, R_2, R_3, Q_4, P_0)$ or $(P_0, Q_1, R_2, R_3', Q_4, P_0)$, where $Q_i \in \Gamma_1$ $(i = 1, 4)$, $R_j \in \Gamma_2^2$ $(j = 2, 3)$, and $R_3' \in \Gamma_2^1$.

LEMMA 5. *Each hexagon in Γ at P_0 is homotopic to a sum of at most two pentagons at P_0.*

PROOF. Let $(P_0, Q_1, R_2, S_3, R_4, Q_5, P_0)$ be a hexagon as above. Then $S_3 = \langle f + x, X_1\rangle$ for some point $x \in PG(5, q)$ which is not contained in any $\langle X_i, X_1\rangle$ $(i \in I - \{1\})$.

When $R_4 \in \Gamma(R_2)$, the given hexagon is homotopic to the pentagon $(P_0, Q_1, R_2, R_4, Q_5, P_0)$ by Lemma 4. Thus we may assume that R_4 is at distance 2 from R_2.

If R_2 and R_4 have the same base, then R_4 belongs to $\Gamma_2^1(R_2)$, the set of points at distance 2 from R_2 with the same base as R_2. Then it follows from Lemma 1 (applied to $\Gamma(R_2) \cap \Gamma(R_4)$ with base point R_2) that there is a point R_7 collinear with S_3, R_2 and R_4 with base X_j for some $j \neq 1$. If R_2 and R_4 have distinct bases, then R_4 belongs to $\Gamma_2^2(R_2)$, the set of points at distance 2 from R_2 with base distinct from that of R_2. Applying Lemma 2 to $\Gamma(R_2) \cap \Gamma(R_4)$, we can also find a point R_7 collinear with S_3, R_2 and R_4 with base X_j for some $j \neq 1$.

In any case, as the base of R_7 is different from X_1, the distance between P_0 and R_7 is at most 2. As R_7 is collinear with $S_3 \in \Gamma_3$, that distance is exactly 2, and there is a point Q_8 in $\Gamma(P_0) \cap \Gamma(R_7)$. Because the triangles (R_7, R_i, S_3, R_7) $(i = 2, 4)$ are null-homotopic by Lemma 4, the given hexagon is homotopic to the sum of two pentagons $(P_0, Q_1, R_2, R_7, Q_8, P_0)$ and $(P_0, Q_8, R_7, R_4, Q_5, P_0)$ based at P_0. □

LEMMA 6. *Each pentagon in Γ at P_0 is homotopic to a sum of two quadrangles at P_0.*

PROOF. Given a pentagon $(P_0, Q_1, R_2, R_3, Q_4, P_0)$ with $Q_i \in \Gamma_1$ $(i = 1, 4)$ and $R_j \in \Gamma_2^2$ $(j = 2, 3)$, we have

$$R_2 = \langle f + x_1 + x_a, X_b\rangle \text{ for some } x_i \in X_i \ (i = 1, a) \text{ with } a \neq b \in I - \{1\}.$$

The base of $R_3 \in \Gamma_2^2$ is X_c for some $c \in I - \{1\}$. We divide the case depending on whether or not $c \in \{a, b\}$. If $c \notin \{a, b\}$, then it follows from our assumption (3) in Subsection 2.1 that the six points X_{ij} $(i \neq j \in \{1, a, b, c\})$ span $PG(5, q)$. Thus

$$x_1 = \alpha_{1a}x_{1a} + \alpha_{1b}x_{1b} + \alpha_{1c}x_{1c} \text{ and } x_a = \beta_{1a}x_{1a} + \beta_{ab}x_{ab} + \beta_{ac}x_{ac}$$

for some $\alpha_{ij}, \beta_{ij} \in \mathbf{F}_q$. As R_2 is collinear with R_3, the point R_3 contains a vector $f + x_1 + x_a + y_b$ for some $y_b \in X_b$. Writing

$$y_b = \gamma_{1b}x_{1b} + \gamma_{ab}x_{ab} + \gamma_{bc}x_{bc}$$

for some $\gamma_{ij} \in \mathbf{F}_q$, the point R_3 with base X_c has the form

$$R_3 = \langle f + (\alpha_{1a} + \beta_{1a})x_{1a} + (\alpha_{1b} + \gamma_{1b})x_{1b} + (\beta_{ab} + \gamma_{ab})x_{ab}, X_c\rangle.$$

As $R_3 \in \Gamma_2^2$, it follows from the last remark in Subsection 2.2 that $\beta_{ab} + \gamma_{ab} \neq 0$.

Now choose a point

$$Q_6 := \langle f + (\alpha_{1b} + \gamma_{1b})x_{1b} + \alpha_{1c}x_{1c}, X_a\rangle.$$

Clearly Q_6 lies in $\Gamma(P_0)$. The point Q_6 is also collinear with both R_2 and R_3. Because the triangle (Q_6, R_2, R_3, Q_6) is null-homotopic by Lemma 4, the given pentagon is homotopic to the sum of two quadrangles $(P_0, Q_1, R_2, Q_6, P_0)$ and $(P_0, Q_6, R_3, Q_4, P_0)$.

If $c \in \{a, b\}$, then $c = a$, as R_3 and R_2 are collinear. Then $f + x_1 + x_a + y_b$ for some $y_b \in X_b$ lies in R_3, and hence $R_3 = \langle f + x_1 + y_b, X_a\rangle$. In this case, we choose an index $c \in I - \{1, a, b\}$, and similarly to the above, write x_1, x_a, y_b as linear combinations of x_{ij}'s. Under the same notation, we can easily verify that the point

$$Q_6 := \langle f + (\alpha_{1a} + \beta_{1a})x_{1a} + (\alpha_{1b} + \gamma_{1b})x_{1b}, X_c\rangle$$

of $\Gamma(P_0)$ is collinear with R_2 and R_3. The claim follows in this case.

It remains to show the claim for a pentagon of the form $(P_0, Q_1, R_2, R_3', Q_4, P_0)$ with $Q_i \in \Gamma_1$ $(i = 1, 4)$, $R_2 \in \Gamma_2^2$ and $R_3' \in \Gamma_2^1$. We have $R_2 = \langle f + x_1 + x_a, X_b\rangle$ and $R_3' = \langle f + y_c, X_1\rangle$ for some $x_i \in X_i$ $(i = 1, a)$ and $y_c \in X_c$, $1 \neq c \in I$. As R_2 is collinear with R_3, $c \neq b$ and $x_a + y_c \in \langle X_1, X_b\rangle$.

When $c \in I - \{1, a, b\}$, we can easily verify that the point $Q_6 := \langle f + x_1 + \alpha_{1a}x_{1a}, X_c\rangle$ is collinear with P_0, R_2 and R_3'. When $c = a$, the point $Q_6 := \langle f + x_1, X_a\rangle$ is collinear with P_0, R_2 and R_3'. □

We now examine the quadrangles in Γ. Observing the bases, each quadrangle at P_0 is either $(P_0, Q_1, R_2, Q_3, P_0)$ or $(P_0, Q_1, R_2', Q_3, P_0)$ for some $Q_1, Q_3 \in \Gamma_1$, $R_2 \in \Gamma_2^2$ and $R_2' \in \Gamma_2^1$. The point R_2 (or R_2') is called the *antipodal point* of the quadrangle (with base point P_0).

LEMMA 7. *Every quadrangle at P_0 is homotopic to a quadrangle at P_0 with specified antipodal point, say $R := \langle f + x_{23}, X_4\rangle$, in Γ_2^2.*

PROOF. We divide the proof into some steps.

STEP 1. Every quadrangle at P_0 with antipodal point $R \in \Gamma_2^2$ is homotopic to a quadrangle with antipodal point $R' \in \Gamma_2^1$.

PROOF. Let $\mathcal{Q}$ be a quadrangle with antipodal point $R_2 = \langle f + x_1 + x_a, X_b\rangle$ in Γ_2^2 $(x_i \in X_i,\ i = 1, a)$. By Subsection 2.3 and Lemma 2, $\mathcal{Q}$ is homotopic to a quadrangle $\mathcal{Q}' := (P_0, Q_1, R_2, Q_3, P_0)$ with $Q_1 = \langle f + \alpha x_{1b}, X_a\rangle$ and $Q_3 = \langle f + \alpha' x_{1b}, X_a\rangle$ for some $\alpha, \alpha' \in \mathbf{F}_q$. Then $R_2' := \langle f + x_a, X_1\rangle$ is a point of Γ_2^1 collinear with R_2, Q_1 and Q_3. Thus, adding two triangles (Q_i, R_2, R_2', Q_i) $(i = 1, 3)$ to $\mathcal{Q}'$, we have a quadrangle $(P_0, Q_1, R_2', Q_3, P_0)$ with antipodal point $R_2' \in \Gamma_2^1$, which is homotopic to the original quadrangle $\mathcal{Q}$.

STEP 2. Let $\langle f + x_a, X_1\rangle$ and $\langle f + x_a', X_1\rangle$ be two distinct points of Γ_2^1 for some $x_a, x_a' \in X_a - X_{1a}$. Assume that X_{1a} is not contained in $\langle x_a, x_a'\rangle$. Then each quadrangle at P_0 with antipodal point $\langle f + x_a, X_1\rangle$ is homotopic to a quadrangle at P_0 with antipodal point $\langle f + x_a', X_1\rangle$.

PROOF. By Subsection 2.3 and Lemma 1, each quadrangle at P_0 with antipodal point $R_2' = \langle f + x_a, X_1\rangle$ is homotopic to a quadrangle $(P_0, Q_1, R_2', Q_3, P_0)$, where $Q_1 = \langle f + x_1, X_a\rangle$ and $Q_3 = \langle f + y_1, X_a\rangle$ for some $x_1, y_1 \in X_1$. Then both Q_1 and Q_3 are collinear with $R_2'' := \langle f + x_a', X_1\rangle$, and $(P_0, Q_1, R_2'', Q_3, P_0)$ is a quadrangle at P_0 with antipodal point R_2''.

As R_2'' is distinct from R_2', $x_a + x_a'$ is not contained in X_1. Thus $[x_a + x_a']$ and X_{1a} are distinct points on the plane X_a, and hence there is a unique line through these two points. Let X_{ab} be the second point on this line in the hyperoval $O_a = \{X_{ai} | i \in I - \{a\}\}$ in X_a. Then $x_a + x_a' \in \langle X_{1a}, X_{ab}\rangle = \langle X_1, X_b\rangle \cap X_a$, and hence the point $R := \langle f + x_1 + x_a, X_b\rangle$ is collinear with R_2''. Clearly R is collinear with Q_1 and R_2'.

Take a point $Q_4(\alpha) := \langle f + x_1 + \alpha x_{1b}, X_a\rangle$ for an arbitrary $\alpha \in \mathbf{F}_q$, which is collinear with P_0, R_2', R and R_2''. In particular, Q_3 and $Q_4(\alpha)$ lie on the common neighbor $\Gamma_1 \cap \Gamma(R_2')$.

(Note that $R_2' \in \Gamma_2^1$; in Subsection 2.3, the points Q_3 and $Q_4(\alpha)$ are denoted by $Q_a'(y_1)$ and $Q_a'(x_1+\alpha x_{1b})$, respectively.) Let c be the index of $I-\{1,a\}$ such that $x_a \in \langle X_{1a}, X_{ac}\rangle$. Note that $b \neq c$, for otherwise $[x_a]$ and $[x_a']$ are two distinct points in the projective line $\langle X_{1a}, X_{ab}\rangle$, which is against our assumption. Then X_{1a}, X_{1b} and X_{1c} are three distinct projective points on a hyperoval in X_1, and hence there is an element $\alpha_0 \in \mathbf{F}_q$ such that $x_1+y_1+\alpha_0 x_{1b} \in \langle X_{1a}, X_{ac}\rangle$. By Lemma 1, in the induced subgraph on $\Gamma_1 \cap \Gamma(R_2')$, the point $Q_c'(y_1) := \langle f+y_1, X_c\rangle$ is collinear with Q_3 and $Q_4(\alpha_0)$.

Hence, the original quadrangle $(P_0, Q_1, R_2', Q_3, P_0)$ with antipodal point R_2' gives rise to a quadrangle $(P_0, Q_1, R_2', Q_4(\alpha_0), P_0)$ by adding four triangles

$$(P_0, Q_3, Q_c'(y_1), P_0),\ (Q_3, R_2', Q_c'(y_1), Q_3),$$
$$(P_0, Q_c'(y_1), Q_4(\alpha_0), P_0) \text{ and } (Q_c'(y_1), R_2', Q_4(\alpha_0), Q_c'(y_1)),$$

which in turn yields a quadrangle $(P_0, Q_1, R_2'', Q_4(\alpha_0), P_0)$ with antipodal point R_2'' by adding the following four triangles:

$$(Q_4(\alpha_0), R_2', R, Q_4(\alpha_0)),\ (Q_4(\alpha_0), R, R_2'', Q_4(\alpha_0)),$$
$$(Q_1, R, R_2'', Q_1) \text{ and } (Q_1, R_2'', R, Q_1).$$

Thus the claim follows.

PROOF OF THE LEMMA. Choose distinct indices $a, b \in I-\{1\}$ and vectors $x_a \in X_a - X_{1a}$, $x_b \in X_b - X_{1b}$. We will first show that each quadrangle with antipodal point $\langle f+x_a, X_1\rangle$ is homotopic to a quadrangle with antipodal point $\langle f+x_b, X_1\rangle$.

Let a' (resp. b') be the index with $x_a \in \langle X_{1a}, X_{aa'}\rangle$ (resp. $x_b \in \langle X_{1b}, X_{bb'}\rangle$). If $a' \neq b$ and $b' \neq a$, then it follows from Step 2 that each quadrangle at P_0 with antipodal point $\langle f+x_a, X_1\rangle$ (resp. $\langle f+x_b, X_1\rangle$) is homotopic to a quadrangle with antipodal point $\langle f+x_{ab}, X_1\rangle$, where x_{ab} is a base of X_{ab}. If $a' = b$, there is an index $c \in I-\{1,a,b,b'\}$ as $|I| = q+3 \geq 5$. Then each quadrangle at P_0 with antipodal point $\langle f+x_a, X_1\rangle$ (resp. $\langle f+x_b, X_1\rangle$) is homotopic to a quadrangle with antipodal point $\langle f+x_{ac}, X_1\rangle$ (resp. $\langle f+x_{bc}, X_1\rangle$). As X_{1c}, X_{ac} and X_{bc} are three distinct points on a hyperoval in X_c, the point X_{1c} is not contained in $\langle x_{ac}, x_{bc}\rangle$. Thus each quadrangle with antipodal point $\langle f+x_{ac}, X_1\rangle$ is homotopic to a quadrangle with antipodal point $\langle f+x_{bc}, X_1\rangle$ by Step 2. When $b' = a$, the same argument can be applied.

By the above claim and Step 1, we conclude that every quadrangle is homotopic to a quadrangle $\mathcal{Q} := (P_0, Q_1, R', Q_3, P_0)$ with a specified antipodal point R' of type Γ_2^1. We may take $R' = \langle f+x_{23}, X_1\rangle$. The common neighbor $\Gamma_1 \cap \Gamma(R')$ consists of the points of the form $\langle f+x_1, X_i\rangle$ for $x_1 \in X_1$ and $i = 2,3$ by Subsection 2.3. Thus, by Step 1, we may take $Q_1 = \langle f+x_1, X_2\rangle$ and $Q_3 = \langle f+y_1, X_2\rangle$ for some $x_1, y_1 \in X_1$. Write $x_1 := \alpha_{12}x_{12}+\alpha_{13}x_{13}+\alpha_{14}x_{14}$ for some $\alpha_{1i} \in \mathbf{F}_q$ ($i = 2,3,4$). The points Q_1, $Q_1'' := \langle f+x_1, X_3\rangle$ and $Q_1' := \langle f+x_1+\alpha_{13}x_{13}, X_2\rangle$ are contained in $\Gamma_1 \cap \Gamma(R')$, and Q_1 and Q_1' are collinear with Q_1''. Replacing $\mathcal{Q}$ by the quadrangle $(P_0, Q_1', R', Q_3, P_0)$ homotopic to $\mathcal{Q}$ if necessary, we may assume that $Q_1 = \langle f+x_1, X_2\rangle$ with $x_1 \in \langle X_{12}, X_{14}\rangle$. Similarly, we may assume that $Q_3 = \langle f+y_1, X_2\rangle$ with $y_1 \in \langle X_{12}, X_{14}\rangle$. Then Q_1 and Q_3 are collinear with the point $R := \langle f+x_{23}, X_4\rangle$, and hence $\mathcal{Q}' := (P_0, Q_1, R, Q_3, P_0)$ is a quadrangle. As R and R' are collinear, $\mathcal{Q}$ is homotopic to the quadrangle $\mathcal{Q}'$ with antipodal point R, and the lemma is established. □

In view of the lemmas of this section, the fundamental group of the geometry $\mathcal{Y}(\mathcal{S})$ (based at P_0) is generated by a quadrangle with specified antipodal point $R := \langle f+x_{23}, X_4\rangle$ of

Γ_2^2. Thus, if the common neighbor $\Gamma_1 \cap \Gamma(R)$ is connected, then every quadrangle is null-homotopic, and hence the fundamental group is trivial. Note that, with the chosen indices $a = 2$ and $b = 4$, the function α_{12} for x_{23} takes the value 0 at the index 3. In this case, the index c of Subsection 2.4 is 3. Thus the following sufficient condition for $\mathcal{Y}(\mathcal{S})$ to be simply connected follows from Lemma 3.

THEOREM 1. *Let $\mathcal{S}$ be a family $\{X_i | i \in I\}$ of $q+3$ planes indexed by I satisfying the three conditions (1),(2),(3) of Subsection 2.1. Choose a base x_{ij} for each X_{ij} ($i \neq j \in I$) and four indices $1, 2, 3, 4 \in I$, and define the functions α_{12} and β_{124} from $I - \{1, 2, 3, 4\}$ to $\mathbf{F}_q^\times$ by the formulae*

$$x_{23} = \alpha_{12}(t)x_{12} + \alpha_{24}(t)x_{24} + \alpha_{2t}x_{2t} \text{ and}$$
$$[\alpha_{12}(t)x_{12} + \beta_{124}(t)x_{14}] = \langle X_{12}, X_{14}\rangle \cap \langle X_{13}, X_{1t}\rangle (t \in I - \{1, 2, 3, 4\}).$$

If the additive subgroup of $\mathbf{F}_q$ generated by $\{\beta_{124}(t) \mid t \in I - \{1, 2, 3, 4\}$ coincides with $\mathbf{F}_q$, then the geometry $\mathcal{Y}(\mathcal{S})$ is simply connected.

4. THE GEOMETRY $\mathcal{Y}_q(K)$ FOR A $(q+1)$-ARC K

For the geometry $\mathcal{Y}_q(K)$, which is obtained from a $(q+1)$-arc K via the Klein correspondence [5, Section 3, Subsection 2.1], we can explicitly compute the functions α_{1a} and hence β_{1ab} in Subsection 2.4 and Theorem 1. It is known that each $(q+1)$-arc in $PG(3, q)$ $(q = 2^e)$ is projectively equivalent to K_m for some $1 \leq m \leq e$ prime to e [2], where K_m consists of the points $P(t) := [1, t, t^{2^m}, t^{2^m+1}]$ for $t \in \mathbf{F}_q$ together with $P(\infty) := [0, 0, 0, 1]$. A transformation $\tau \in PGL_4(q)$ maps $\mathcal{S}(K)$ onto $\mathcal{S}(K^\tau)$, whence it induces an isomorphism of the geometries $\mathcal{Y}(\mathcal{S}(K))$ and $\mathcal{Y}(\mathcal{S}(K^\tau))$. Thus, we may assume that $K = K_m$ for some $1 \leq m \leq e$ prime to e. For simplicity, we denote by σ the Galois automorphism of $\mathbf{F}_q/\mathbf{F}_2$ defined by $t^\sigma = t^{2^m}$.

The two systems of generators of K_m are given as $\mathcal{M} := \{m_t | t \in \{\infty\} \cup \mathbf{F}_q\}$ and $\mathcal{N} := \{n_t | t \in \{\infty\} \cup \mathbf{F}_q\}$, where m_t and n_t are the following lines intersecting at $P(t)$:

$$m_t := \{[x, y, xt^\sigma, yt^\sigma] \mid x, y \in \mathbf{F}_q\}, m_\infty := \{[0, 0, x, y] \mid x, y \in \mathbf{F}_q\},$$
$$n_t := \{[x, xt, y, yt] \mid x, y \in \mathbf{F}_q\}, n_\infty := \{[0, x, 0, y] \mid x, y \in \mathbf{F}_q\} \text{ for } t \in \mathbf{F}_q.$$

We recall that the Klein correspondence θ is a map sending each line $\langle \mathbf{a}, \mathbf{b}\rangle$ of $PG(3, q)$ to a point $(c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34})$ of $PG(5, q)$, where $c_{ij} := a_i b_j - a_j b_i$ for $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $\mathbf{b} = (b_1, b_2, b_3, b_4)$ $(1 \leq i < j \leq 4)$. The image of θ is the set $Q^+(5, q)$ of zeros of the quadratic form $x_1x_6 + x_2x_5 + x_3x_4$ on $PG(5, q)$. The set of lines through a point P of $PG(3, q)$ is mapped onto the set of points on a totally singular plane of $Q^+(5, q)$, which is denoted by $\theta(P)$.

The singular plane $X(t) := \theta(P(t))$ for the $q+1$ points $P(t)$ on K_m is as follows:

$$X(t) = \{[a, b, c, at^\sigma + bt, at^{\sigma+1} + ct, bt^{\sigma+1} + ct^\sigma] \mid a, b, c \in \mathbf{F}_q\} \text{ for } t \in \mathbf{F}_q,$$

in particular,

$$X(0) = \{[a, b, c, 0, 0, 0] \mid a, b, c \in \mathbf{F}_q\},$$
$$X(\infty) := \theta(P(\infty)) = \{[0, 0, c, 0, e, f] \mid c, e, f \in \mathbf{F}_q\} \text{ for } t \in \mathbf{F}_q.$$

Furthermore, the images of the lines in each one of the two systems $\mathcal{M}$ and $\mathcal{N}$ above generate the following (non-singular) planes:

$$X(\mathcal{M}) := \langle\theta(m_t)|m_t \in \mathcal{M}\rangle = \{[x, 0, y, y, 0, z] \mid x, y, z \in \mathbf{F}_q\},$$
$$X(\mathcal{N}) := \langle\theta(n_t)|n_t \in \mathcal{N}\rangle = \{[0, x, y, y, z, 0] \mid x, y, z \in \mathbf{F}_q\}.$$

It is not difficult to verify that the family $\mathcal{S}(K_m) := \{X(t)|t \in \{\mathcal{N}, \mathcal{M}, \infty\} \cup \mathbf{F}_q\}$ of the above $q+3$ planes satisfies the conditions (1)–(3) in Subsection 2.1. Thus we can apply all the results in the earlier sections to the extended generalized quadrangle $\mathcal{Y}_q(K_m) := \mathcal{Y}(\mathcal{S}(K_m))$ constructed from the family $\mathcal{S}(K_m)$.

In order to determine the fundamental group of $\mathcal{Y}_q(K_m)$, we will apply Theorem 1 to the following four planes

$$X_1 := X(\mathcal{N}),\ X_2 := X(\mathcal{M}),\ X_3 := X(\infty) \text{ and } X_4 := X(0).$$

We choose the following vectors x_{ij} as bases for the points $X_{ij} = X_i \cap X_j$ $(1 \leq i < j \leq 4)$:

$$x_{12} = (0, 0, 1, 1, 0, 0),\ x_{13} = (0, 0, 0, 0, 1, 0),\ x_{14} = (0, 1, 0, 0, 0, 0),$$
$$x_{23} = (0, 0, 0, 0, 0, 1),\ x_{24} = (1, 0, 0, 0, 0, 0),\ x_{34} = (0, 0, 1, 0, 0, 0).$$

Furthermore, for each $t \in \mathbf{F}_q^{\times}$ (corresponding to the subset $I - \{1, 2, 3, 4\}$ of indices), the points $X_{1t} = X_1 \cap X(t)$ and $X_{2t} = X_2 \cap X(t)$, respectively, have bases

$$x_{1t} = (0, 1, t, t, t^2, 0) \text{ and } x_{2t} = (1, 0, t^{\sigma}, t^{\sigma}, 0, t^{2\sigma}).$$

We will now compute the functions α_{1a} and β_{1ab} defined in Subsection 2.4 for $a = 2$, $b = 4$. We have $x_{23} = t^{-\sigma}x_{12} + t^{-2\sigma}x_{24} + t^{-2\sigma}x_{2t}$ for $t \in \mathbf{F}_q^{\times}$. Thus the function α_{12} is given by $\alpha_{12}(t) = t^{-\sigma}$ for $t \in \mathbf{F}_q^{\times}$, and $\alpha_{12}(3) = 0$ (that is, the index c is 3 in this case). As $\langle X_{13}, X_{1t}\rangle = \{(0, b, bt, bt, bt^2 + c, 0)|b, c \in \mathbf{F}_q\}$ for $t \in \mathbf{F}_q^{\times}$, the projective point

$$[\alpha_{12}(t)x_{12} + \beta_{124}(t)x_{14}] = [0, \beta_{124}(t), t^{-\sigma}, t^{-\sigma}, 0, 0]$$

lies in $\langle X_{13}, X_{1t}\rangle$ exactly when $\beta_{124}(t) = t^{-(\sigma+1)}$.

The following lemma shows that the values $t^{-(\sigma+1)}$ $(t \in \mathbf{F}_q^{\times})$ of the function β_{124} generate the whole additive group $\mathbf{F}_q$, except when $q = 4$.

LEMMA 8. *Let $\mathbf{F}_q$ be a finite field of cardinality $q = 2^e$, and let m be a integer prime to e with $1 \leq m \leq e-1$. If $q = 4$, then*

$$\mathbf{F}_q = \{\alpha^{2^m+1} + \beta^{2^m+1} \mid \alpha, \beta \in \mathbf{F}_q\}.$$

PROOF. Note first that the greatest common divisor $(2^m + 1, q - 1)$ is 1 or 3, and that the latter holds exactly when m is odd and e is even.

Let $Q_m := \{\alpha^{2^m+1} \mid \alpha \in \mathbf{F}_q\}$ be the set of (2^m+1)-th powers in $\mathbf{F}_q$. The set $Q_m^{\times} = Q_m - \{0\}$ is a subgroup of the cyclic group $\mathbf{F}_q^{\times}$ of index $(2^m + 1, q - 1)$. In particular, $Q_m^{\times} = \mathbf{F}_q^{\times}$ for $q = 2^e$ with e odd or m even, and the claim follows in this case.

Let e be even and m odd. Then Q_m coincides with the set of cubes of $\mathbf{F}_q$. In this case, when $q \neq 4$, every element of $\mathbf{F}_q$ is the sum of two cubes. □

Thus it follows from Theorem 1 that $\mathcal{Y}_q(K_m)$ is simply connected, if $q \neq 4$.

When $q = 4$, the values of the function β_{124} generate the subfield $A = \mathbf{F}_2$ of index 2 in the additive group $\mathbf{F}_4$. In fact, there are exactly two connected components in the induced subgraph on $\Gamma_1 \cap \Gamma(R)$. Thus the fundamental group is generated by the homotopy class of the quadrangle $(P_0, Q_2(0), R, Q_2(\omega), P_0)$, where ω is an element of $\mathbf{F}_4 - \mathbf{F}_2$. To determine the universal cover in this case, we will argue as follows: note that $m = 1$ as $q = 4$, and that $\mathcal{Y}_4(K_1)$ is isomorphic to the extended generalized quadrangle $\mathcal{Y}_4(O)$ obtained from the regular hyperoval O in $PG(2, 4)$ (see Subsection 2.1). The latter is known to be flag-transitive by [7, Section 4]. Using generators and relations, the existence of a double cover of $\mathcal{Y}_4(O)$ together with its simple connectedness is shown in [8].

Summarizing, we have established:

THEOREM 2. *For the family $\mathcal{S}(K)$ of planes constructed in [5] from a $(q + 1)$-arc K in $PG(3, q)$ $(q = 2^e)$, the extended generalized quadrangle $\mathcal{Y}_q(\mathcal{S}(K))$ is simply connected except when $q = 4$. In the exceptional case, the universal cover of $\mathcal{Y}_4(\mathcal{S}(K))$ is a double cover.*

REMARK. A double cover of $\mathcal{Y}_4(K_1)$ can be constructed inside the unitary polar space $H(7, 4)$ as follows:

As we can see in [1, p.39], there is a set $\mathcal{M}$ of 22 isotropic planes in $H(5, 4)$ such that any two planes of $\mathcal{M}$ have just one point in common. The stabilizer in $SU_6(4)$ of $\mathcal{M}$ induces the permutation group on $\mathcal{M}$ equivalent to the Mathieu group M_{22} on the Steiner system $S(22, 6, 3)$. We can verify that a subset $\mathcal{S}$ of $\mathcal{M}$ corresponding to a heptad satisfies the properties (1)–(3) in Subsection 2.1. Thus $\mathcal{S}(K_1)$ are realized as $\mathcal{S}$ in $H(5, 4)$.

Let p be a point of $H(7, 4)$ and let Δ be the affine polar space obtained by removing the hyperplane $p^\perp$ from $H(7, 4)$. The group of elations with center p and axis $\langle p^\perp \rangle$ intersects $Aut(H(7, 4))$ in a group E of order 2. The complement $\mathcal{A}$ of the star $\langle p^\perp \rangle / p \cong PG(5, 4)$ in the star $PG(7, 4)/p \cong PG(6, 4)$ is an affine space $AG(6, q)$. Note that every line of $PG(7, 4)$ through p not contained in $\langle p^\perp \rangle$ intersects Δ in exactly two points distinct from p, which form an orbit of E. Thus the points of $\mathcal{A}$ bijectively correspond to the E-orbits on the set of points of Δ.

On the other hand, under the identification of $H(5, 4)$ with $p^\perp / p$, the points, lines and planes of $\mathcal{Y}_4(K_1)$ are realized as the points of $\mathcal{A}$, the lines with point at infinity belonging to a member of $\mathcal{S}(K_1)$ ($\subset H(5, 4)$, as we remarked above), and the 3-spaces of $\mathcal{A}$ with a member of $\mathcal{S}(K_1)$ as the plane at infinity. Thus, if we take the inverse images by E of these objects, then we have a subgeometry $\tilde{\mathcal{Y}}_4(K_1)$ of Δ. It can be immediately seen that E defines the quotient of $\tilde{\mathcal{Y}}_4(K_1)$ and $\tilde{\mathcal{Y}}_4(K_1)/E \cong \mathcal{Y}_4(K_1)$. Thus $\tilde{\mathcal{Y}}_4(K_1)$ gives a double cover of $\mathcal{Y}_4(K_1)$.

5. THE GEOMETRY $\mathcal{Y}_q(O)$ FOR A HYPEROVAL O IN $PG(2, q)$

Any transformation $\tau \in PGL(3, q)$ maps the family $\mathcal{S}(O)$ onto $\mathcal{S}(O^\tau)$, whence it induces an isomorphism of $\mathcal{Y}_q(O)$ with $\mathcal{Y}_q(O^\tau)$. Thus to examine the universal cover of $\mathcal{Y}_q(O)$, we may replace the hyperoval O by any hyperoval projectively equivalent to it.

Each hyperoval in $PG(2, q)$ is projectively equivalent to the hyperoval $O(f)$ consisting of the following $q + 2$ points $u(t)$ $(t \in \mathbf{F}_q)$, $u(\infty)$ and $u(n)$;

$$u(t) := [1, t, f(t)] (t \in \mathbf{F}_q), u(\infty) := [0, 0, 1], \text{ and } u(n) := [0, 1, 0],$$

for some permutation polynomial f over $\mathbf{F}_q$ with $f(0) = 0$.

The dual of $O(f)$ consists of the following lines:

$$l(t) := \{(x_0, x_1, x_2) \mid x_0 + tx_1 + f(t)x_2 = 0\} (t \in \mathbf{F}_q),$$
$$l(\infty) := \{(x_0, x_1, 0) | x_0, x_1 \in \mathbf{F}_q\}, \text{ and } l(n) := \{(x_0, 0, x_2) | x_0, x_2 \in \mathbf{F}_q\}.$$

It can be seen immediately that the images of these lines via the Veronesean map

$$\zeta : PG(2,q) \ni [x_0, x_1, x_2] \mapsto [x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2] \in PG(5,q)$$

lie on the following planes (called *conic planes*).

$$\Pi(t) := \left\{ (y_{00}, y_{11}, y_{22}, y_{01}, y_{02}, y_{12}) \;\middle|\; \begin{array}{l} y_{00} + ty_{01} + f(t)y_{02} = 0 \\ y_{01} + ty_{11} + f(t)y_{12} = 0 \\ y_{02} + ty_{12} + f(t)y_{22} = 0 \end{array} \right\}, \quad t \in \mathbf{F}_q,$$

$$\Pi(\infty) := \{(y_{00}, y_{11}, 0, y_{01}, 0, 0) | y_{00}, y_{11}, y_{01} \in \mathbf{F}_q\},$$

$$\Pi(n) := \{(y_{00}, 0, y_{22}, 0, y_{02}, 0) | y_{00}, y_{22}, y_{02} \in \mathbf{F}_q\}.$$

Adding the nucleus plane $N := \{(0, 0, 0, y_{01}, y_{02}, y_{12}) | y_{01}, y_{02}, y_{12} \in \mathbf{F}_q\}$ to these $q+2$ planes, we obtain a family $\mathcal{S}(O(f)) := \{X_i | i \in I\}$ of $q+3$ planes from which we can construct the extended generalized quadrangle $\mathcal{Y}_q(O(f)) := \mathcal{Y}(\mathcal{S}(O(f)))$ as in [7, Section 2]. We index the family $\mathcal{S}(O(f))$ as follows:

$$X_1 := N,\ X_2 := \Pi(n),\ X_3 := \Pi(\infty),\ X_4 := \Pi(0);$$

and

$$X(t) := \Pi(t) \text{ for } t \in \mathbf{F}_q^\times.$$

From the explicit shapes of these planes, the vectors

$$\begin{aligned} x_{12} &:= (0,0,0,0,1,0),\ x_{23} := (1,0,0,0,0,0),\\ x_{13} &:= (0,0,0,1,0,0),\ x_{24} := (0,0,1,0,0,0),\\ x_{14} &:= (0,0,0,0,0,1),\ \text{ and } x_{34} := (0,1,0,0,0,0) \end{aligned}$$

are bases for projective points X_{ij} ($1 \le i \ne j \le 4$), respectively; and $X_{2t} := X_2 \cap X(t)$ and $X_{1t} := X_1 \cap X(t)$ have bases

$$x_{2t} := (f(t)^2, 0, 1, 0, f(t), 0),\ x_{1t} := (0,0,0,f(t),t,1) \text{ for } t \in \mathbf{F}_q^\times, \text{ respectively.}$$

Thus for each $t \in \mathbf{F}_q^\times$, we have $x_{2t} = f(t)^2 x_{23} + x_{24} + f(t)x_{12}$, and hence

$$x_{23} = \alpha_{12}(t)x_{12} + f(t)^{-2}x_{24} + f(t)^{-2}x_{2t}, \text{ where } \alpha_{12}(t) = f(t)^{-1}.$$

Furthermore, we have

$$\langle X_{12}, X_{14}\rangle = \{(0,0,0,0,y_{02},y_{12}) | y_{02}, y_{01} \in \mathbf{F}_q\}$$

and

$$\langle X_{13}, X_{1t}\rangle = \{(0,0,0,\gamma + \delta f(t), \delta t, \delta) | \gamma, \delta \in \mathbf{F}_q\},$$

and hence $\langle X_{12}, X_{14}\rangle \cap \langle X_{13}, X_{1t}\rangle = [0,0,0,0,t,1]$. From the defining formula $[0,0,0,0,t,1] = [\alpha_{12}(t)x_{12} + \beta_{124}(t)x_{14}]$, we obtain $\beta_{124}(t) = (tf(t))^{-1}$ for $t \in \mathbf{F}_q^\times$.

To apply Theorem 1, we have to consider the additive subgroup of $\mathbf{F}_q$ generated by $\beta_{124}(t)^{-1} = tf(t)$ for each choice of f.

According to the list in [6, p. 299], there are six infinite families of hyperovals $O(f)$ for which the explicit shapes of f are known. They are (S.i) by Segre ($i = 1, 2$), (G.i) by Glynn ($i = 1, 2, 3$), and (P) by Payne.

(S.1) $f(t) = t^{2^m}$ for $1 \le m \le e-1$ prime to e,
(S.2) $f(t) = t^6$ with e odd,
(G.1) $f(t) = t^{3.2^n+4}$ for $e = 2n-1$,
(G.2) $f(t) = t^{2^{m+1}+2^{3m+1}}$ for $e = 4m+1$,
(G.3) $f(t) = t^{2^{m+1}}$ for $e = 4m-1$,
(P) $f(t) = t^{1/6} + t^{1/2} + t^{5/6}$ for e odd.

The polynomials f are also given for three sporadic hyperovals (C) by Cherowitzo. The polynomial (LS) for the Lunelli–Sce hyperoval is given in [3, 8.4].

(C) $f(t) = t^8 + t^{10} + t^{28}$ for $q = 2^5$, $f(t) = t^{16} + t^{18} + t^{52}$ for $q = 2^7$, and $f(t) = t^{32} + t^{34} + t^{100}$ for $q = 2^9$.
(LS) $f(t) = (\eta^2 t^7 + \eta^{12} t^6 + \eta^6 t^5 + \eta^9 t^4 + \eta^5 t^3 + \eta^5 t^2 + \eta^6 t)^2$ for $q = 16$, where $\eta \in \mathbf{F}_{16}$ with $\eta^4 = \eta + 1$.

We will examine whether or not the values of $tf(t)$ generate the whole $\mathbf{F}_q$ for each of the above polynomials f. (The author does not claim that the above hyperovals are all the currently known classes of hyperovals.)

It is tedious, but straightforward, to verify that the values of $tf(t)$ for $t \in \mathbf{F}_q$ generate the whole additive group $\mathbf{F}_q$ in each of the sporadic cases. The case (S.1) is settled by Lemma 8 in the previous section. For the cases (S.2), (G.i) ($i = 1, 2, 3$), we may apply the following lemma. The proof is left for the reader, as it is similar to that of Lemma 8.

LEMMA 9. *Let $\mathbf{F}_q$ be a finite field of cardinality $q = 2^e$.*

(1) *Except when $q = 8$, the additive group $\mathbf{F}_q$ is generated by α^7, α ranging over $\mathbf{F}_q$.*
(2) *If $e = 2n-1$, the additive group $\mathbf{F}_q$ is generated by $\alpha^{3.2^n+5}$, α ranging over $\mathbf{F}_q$.*
(3) *If $e = 4m+1$, the additive group $\mathbf{F}_q$ is generated by $\alpha^{2^{2m+1}+2^{3m+1}+1}$, α ranging over $\mathbf{F}_q$.*
(4) *If $e = 4m-1$, the additive group $\mathbf{F}_q$ is generated by $\alpha^{2^{m+1}+1}$, α ranging over $\mathbf{F}_q$.*

We do not worry about the exceptional case $q = 8$ of Lemma 9(1), because, as every hyperoval in $PG(2, 8)$ is regular (e.g., [3, 8.4.3]), this case has been settled in the previous section (or in the case (S.1)). Hence the following results follow from Theorem 1.

THEOREM 3. *Let $\mathcal{S}(O)$ be the family of planes constructed in [7] from a hyperoval O in $PG(2, q)$ ($q = 2^e$), and let $\mathcal{Y}_q(O)$ be the extended generalized quadrangle obtained from $\mathcal{S}(O)$. Assume that $O = O(f)$ is determined by a polynomial f as in [6, p.299]. Then $\mathcal{Y}_q(O)$ is simply connected except when $q = 4$ and possibly when f belongs to the family of Payne.*

REMARKS. (1) As for the polynomial $f(t) = t^{1/6} + t^{1/2} + t^{1/6}$ over $\mathbf{F}_{2^e}$ with e odd, the author is presently unable to show that the values of $tf(t)$ always generate the whole additive group $\mathbf{F}_{2^e}$, although that is likely the case.

(2) When $q = 8$, in the above presentation of $\mathcal{Y}_8(O(f))$ with $f(t) = t^7$, the common neighbor of $P_0 = \langle f, X_1 \rangle$ and $\langle f + x_{23}, X_4 \rangle$ splits into four connected components, and we cannot apply Theorem 1. This implies that there is another point at distance 2 from P_0 with a connected common neighbor. Thus, in general, the efficiency of Theorem 1 depends on our choice of the antipodal point and the four planes X_i ($i = 1, 2, 3, 4$) which generate $PG(5, q)$.

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